5) $\bar{x}$ = 0.5; 6) $\bar{x}$ = 0.55; 7) $\bar{x}$ = 0.65], while the velocity distribution in the jet part of the wall jet (not presented in the article) retains the form of the velocity profile of that part of the jet in the mixing zone which turns into the cavity after hitting the back wall (below the line y = 0), i.e., exactly the same pattern as in a square cavity (Fig. 3 of [3]) is observed.

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FLOW OF A LIQUID FILM OVER THE INNER SURFACE OF A ROTATING CYLINDER

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1. The dimensionless equations of motion and continuity and the boundary conditions in a coordinate system y, z, φ (where $z = z^\circ$ is the axial coordinate, $\varphi = \varphi^\circ$, $y = R - r^\circ$, the origin is located on the joint between semiinfinite tubes, z°, r°, φ° is a cylindrical coordinate system) rotating about the axis of symmetry of a cylinder with the angular velocity of rotation of the upper semiinfinite tube have the form [1, 2]

$$\varepsilon\left(v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}+\frac{v_\theta^2}{\eta+y}\right)+2v_\theta=-\frac{\partial p}{\partial y}+E\left[\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}-\frac{1}{\eta+y}\frac{\partial v_y}{\partial y}-\frac{v_y}{(\eta+y)^2}\right], \quad (1.1)$$

$$\varepsilon\left(v_y\frac{\partial v_\theta}{\partial y}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_yv_\theta}{\eta+y}\right)+2v_y=E\left[\frac{\partial^2 v_\theta}{\partial y^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{1}{\eta+y}\frac{\partial v_\theta}{\partial y}-\frac{v_\theta}{(\eta+y)^2}\right],$$

$$\varepsilon\left(v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+E\left(\frac{\partial^2 v_z}{\partial z^2}+\frac{\partial^2 v_z}{\partial y^2}-\frac{1}{\eta+y}\frac{\partial v_z}{\partial y}\right)+\frac{1}{\mathrm{Fr}},$$

$$\frac{\partial v_z}{\partial z}+\frac{\partial v_y}{\partial y}+\frac{v_y}{\eta+y}=0,\quad v_y=-v_r;$$

$$y=0,\ z<0,\ v_y=v_z=v_\theta=0, \quad (1.2)$$

$$z>0,\ v_y=v_z=0,\ v_\theta=\omega\eta;$$

$$y=h(z),\ \left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right)(1+h_z^2)+4h_z\frac{\partial v_y}{\partial y}=0, \quad (1.3)$$

$$2E\frac{\partial v_y}{\partial y}(1-h_z^2)-Eh_z\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial y}\right)-\frac{\alpha}{(\eta+y)^2}[h^2+(\eta+y)^2h_{zz}](1+h_z^2)=$$

$$=p(1+h_z^2),\quad v_y=h_zv_z,\quad \partial v_\theta/\partial y+v_\theta/(\eta+y)=0.$$

Boundary conditions as $z\to\pm\infty$ will be described below. The problem of Eqs. (1.1)-(1.3) is reduced to dimensionless form by replacement of the variables r, v, Ω, p by their

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normalized values rh', vU', $\Omega_1'k$, $\rho'\Omega'_1U'h'p$. The dimensioned values are denoted with primes as follows:

$$h' = (3Q'\nu'/g')^{1/3}, \quad Q' = Q_0'/2\pi R', \quad \eta = R'/h',$$
$$U' = Q'/h' = Q'(3Q'\nu'/g')^{-1/3}, \quad v_\lambda = v_\lambda'/U' \quad (\lambda = y, z, \theta),$$
$$p = (p' - p_0')/(\rho'\Omega_1'U'h'), \quad h = h'(z)/h',$$
$$\omega = \omega/\Omega_1', \quad \Omega_2 = 1 + \omega, \tag{1.4}$$

where $E = \nu'/\Omega_1'h'^2$ is the Eckman number, $\varepsilon = U'/\Omega_1'h'$ is the Rossby number, $Fr = \Omega_1'U'/g'$ is the Froud number, $\alpha = \sigma'/(\rho'U'\Omega_1'h)$; σ' is the surface tension coefficient; v_θ, v_z, v_y are the velocity components along the axes θ, z, y, Ω_1' is the angular velocity of the upper semiinfinite tube, ω is the distance between the angular velocities of the upper tube Ω_1' and the lower tube Ω_2', p_0' is the pressure in the gas phase, ν' is the kinematic viscosity, g' is the gravitational constant, Q_0' is the liquid flow rate.

Boundary conditions (1.2) describe adhesion on the walls of the upper and lower tubes. Boundary conditions (1.3) equate the tangent stress on the film surface to zero and the normal stress to the pressure of the gas phase. Conditions (1.3) are written with consideration of the curvature of the film surface.

We will study the case in which inertial forces (except the Coriolis force) may be neglected, i.e., the case in which the following expressions are satisfied: $\varepsilon/E \ll 1$, $Fr^{-1}/E \sim O(1)$, $E \sim O(1)$. Substituting Eq. (1.4) in these equations, we obtain

$$\varepsilon/E = Q'/\nu' \ll 1, \ \mathrm{Fr}^{-1}/E = 3 \sim O(1),$$
$$E = \nu'(\Omega_1')^{-1}(3Q'\nu'/g')^{-2/3} \sim O(1). \tag{1.5}$$

If conditions (1.5) are satisfied, then all terms in Eq. (1.1) (except inertial ones) are of the same magnitude, so that for the last expression to be satisfied it is necessary that $\Omega_1' \sim \nu' (3Q'\nu'/g)^{-2/3}$, since $Q'/\nu' \ll 1$, while $g' = 10^3$ cm^2/sec, so that $\Omega_1' \gg 1$. A detailed analysis of the forces produced by liquid motion in rotating channels is presented in [1].

We will consider the case in which the difference between the angular velocities of the tubes is small, i.e., $\omega \ll 1$. We relate the small parameter ε to ω bt the expression $\omega = \varepsilon\Omega_0$ ($\Omega_0 = \Omega_0'/\Omega_1'$), $\Omega_0 \sim O(1)$. We will consider flow in a tube the radius of which $\eta \sim O(\varepsilon^{-1/2})$. For convenience in evaluating the terms of Eq. (1.1) we introduce the new parameter $R = \varepsilon^{1/2}\eta$. In this case Eq. (1.2) takes on the form

$$y = 0, \ z > 0, \ v_\theta = \varepsilon^{1/2}\Omega_0 R.$$

To solve the problem of Eqs. (1.1)-(1.3) it is necessary to find the velocity profiles and pressure distribution as $z \to \pm\infty$. To do this we consider separately the flows in region I, located infinitely ($z \to -\infty$) up the flow, and in region II, located infinitely ($z \to +\infty$) down the flow.

2. In regions I and II the flow is independent of z (derivatives with respect to z are equal to zero, there is no radial flow, film thickness is constant). Moreover, since one and the same force acts in the axial plane in regions I and II, the profiles of the axial velocity component and liquid film thickness are identical. In regions I and II the problem of Eqs. (1.1)-(1.3) reduces to the form

$$2v_\theta = -\frac{\partial p}{\partial y}, \quad \frac{\partial v_z}{\partial z} = 0, \quad \frac{\partial^2 v_\theta}{\partial y^2} - \frac{\varepsilon^{1/2}}{R}\frac{\partial v_\theta}{\partial y} - \frac{v_\theta \varepsilon}{R^2} = 0,$$
$$-\frac{\partial p}{\partial z} + E\left(\frac{\partial^2 v_z}{\partial y^2} - \frac{\varepsilon^{1/2}}{R}\frac{\partial v_z}{\partial y}\right) + \frac{1}{\mathrm{Fr}} = 0; \tag{2.1}$$

$$y = 0,\ v_y = v_z = v_\theta = 0; \tag{2.2}$$

$$y = 0,\ v_y = v_z = 0,\ v_\theta = \varepsilon^{1/2} R\Omega_0; \tag{2.3}$$

$$y = 1,\ \partial v_\theta/\partial y + v_\theta \varepsilon^{1/2}/R = 0,\ \partial v_z/\partial y = 0,\ \alpha\varepsilon/R^2 + p = 0. \tag{2.4}$$

Here condition (2.2) is written for region I, and condition (2.3) for region II. We will seek a solution of the problems of Eqs. (2.1), (2.2), (2.4) and Eqs. (2.1), (2.3), (2.4) in the form

$$v_\lambda = v_\lambda^0 + \varepsilon^{1/2} v_\lambda^{(1)} + \varepsilon v_\lambda^{(2)} + \ldots\ (\lambda = z, \theta), \quad p = p^0 + \varepsilon^{1/2} p^{(1)} + \varepsilon p^{(2)} + \ldots \tag{2.5}$$

Substituting Eq. (2.5) in Eqs. (2.1)-(2.4), we find the solution of the problem in regions I and II:

$$v_z = \frac{1}{E\,\mathrm{Fr}}\left(-\frac{y^2}{2} + y\right) + \varepsilon^{1/2} \frac{1}{RE\,\mathrm{Fr}}\left(-\frac{y^3}{6} + \frac{y^2}{2} - \frac{y}{2}\right); \tag{2.6}$$

$$p = 0,\ v_\theta = 0,\ h = 1; \tag{2.7}$$

$$p = 2\varepsilon^{1/2}(y - 1)\Omega_0 - 2\varepsilon[\Omega_0(y^2 - 1)/2 - \alpha/R^2], \quad v_\theta = \varepsilon^{1/2} R\Omega_0 - \varepsilon y\Omega_0,\ h = 1. \tag{2.8}$$

Thus, the solution in region I has the form of Eqs. (2.6), (2.7), while the solution in region II is given by Eqs. (2.6), (2.8). Consequently, the solutions of (1.1)-(1.3) for the tube junction region must tend to Eqs. (2.6), (2.7) as $z \to -\infty$, and to Eqs. (2.6), (2.8) as $z \to +\infty$.

3. We will consider the solution in the junction region. We will seek a solution of Eqs. (1.1)-(1.3), (2.6), (2.8) in the form

$$v_\lambda = v_\lambda^{(0)} + \varepsilon^{1/2} v_\lambda^{(1)} + \varepsilon v_\lambda^{(2)} + \ldots\ (\lambda = y, z, \theta), \quad p = p^{(0)} + \varepsilon^{1/2} p^{(1)} + \varepsilon p^{(2)} + \ldots,\ h = 1 + \varepsilon^{1/2} h^{(1)} + \varepsilon h^{(2)} + \ldots \tag{3.1}$$

Substituting Eq. (3.1) in Eqs. (1.1)-(1.3), (2.6), (2.8), we obtain

$$2v_\theta^{(0)} = -\frac{\partial p^{(0)}}{\partial y} + E\left(\frac{\partial^2 v_y^{(0)}}{\partial y^2} + \frac{\partial^2 v_y^{(0)}}{\partial z^2}\right), \quad 2v_y = E\left(\frac{\partial^2 v_\theta^{(0)}}{\partial y^2} + \frac{\partial^2 v_\theta^{(0)}}{\partial z^2}\right),$$
$$0 = -\frac{\partial p^{(0)}}{\partial z} + E\left(\frac{\partial^2 v_z^{(0)}}{\partial y^2} + \frac{\partial^2 v_z^{(0)}}{\partial z^2}\right) + \frac{1}{\mathrm{Fr}},$$
$$\frac{\partial v_z^{(0)}}{\partial z} + \frac{\partial v_y^{(0)}}{\partial y} = 0; \tag{3.2}$$

$$y = 0,\ v_y^{(0)} = v_z^{(0)} = v_\theta^{(0)} = 0; \tag{3.3}$$

$$y = 1,\ 2E\frac{\partial v_y}{\partial y} = p^{(0)},\ \frac{\partial v_y^{(0)}}{\partial z} + \frac{\partial v_z^{(0)}}{\partial y} = 0,\ \frac{\partial v_\theta^{(0)}}{\partial y} = 0; \tag{3.4}$$

$$z \to \pm\infty,\ v_y^{(0)} \to 0,\ v_z^{(0)} \to \frac{1}{E\,\mathrm{Fr}}\left(-\frac{y^2}{2} + y\right),\ v_\theta^{(0)} \to 0. \tag{3.5}$$

By writing the axial velocity in the form $v_z = \frac{1}{E\,\mathrm{Fr}}\left(-\frac{y^2}{2} + y\right) + V_*$ we reduce the equations and boundary conditions (3.2)-(3.5) to homogeneous equations and boundary conditions for the functions $p^{(0)}, V_*, v_y^{(0)}, v_\theta^{(0)}$. The solutions of these equations with the boundary conditions found are identically equal to zero. Thus, the solution of Eqs. (3.2)-(3.5) has the form

$$v_z^{(0)} = \frac{1}{E\,\mathrm{Fr}}\left(-\frac{y^2}{2} + y\right),\ v_y^{(0)} = v_\theta^{(0)} = p^{(0)} = 0. \tag{3.6}$$

We will now find the first approximation. Since ε appears implicitly in the argument of the zeroth approximation function in the boundary conditions at y = 1, it is necessary, according to [3, 4], first to expand the zeroth approximation function in a Taylor series near y = 1. Then Eqs. (1.1)-(1.3) take on the form

$$2v_\theta^{(1)} = -\frac{\partial p^{(1)}}{\partial y} + E\left(\frac{\partial^2 v_y^{(1)}}{\partial y^2} + \frac{\partial^2 v_y^{(1)}}{\partial z^2}\right), \tag{3.7}$$

$$2v_y^{(1)} = E\left(\frac{\partial^2 v_\theta^{(1)}}{\partial y^2} + \frac{\partial^2 v_\theta^{(1)}}{\partial z^2}\right),$$

$$0 = -\frac{\partial p^{(1)}}{\partial z} + E\left(\frac{\partial^2 v_z^{(1)}}{\partial z^2} + \frac{\partial^2 v_z^{(1)}}{\partial y^2}\right) - \frac{1}{R\,\mathrm{Fr}}(-y+1),$$

$$\frac{\partial v_y^{(1)}}{\partial y} + \frac{\partial v_z^{(1)}}{\partial z} = 0;$$

$$y = 0,\quad z < 0,\quad v_y^{(1)} = v_z^{(1)} = v_\theta^{(1)} = 0,$$
$$z > 0,\quad v_y^{(1)} = v_z^{(1)} = 0,\quad v_\theta^{(1)} = \Omega_0 R; \tag{3.8}$$

$$y = 1,\quad 2E\left(\frac{\partial v_y^{(1)}}{\partial y} + h^{(1)}\frac{\partial^2 v_y^{(0)}}{\partial y^2}\right) + \alpha h_{zz}^{(1)} + h_z^{(1)}\left(\frac{\partial v_y^{(0)}}{\partial z} + \frac{\partial v_z^{(0)}}{\partial y}\right)E =$$
$$= p^{(1)} + h^{(1)}\frac{\partial p^{(0)}}{\partial y},\quad \frac{\partial v_z^{(1)}}{\partial y} + \frac{\partial v_y^{(1)}}{\partial z} + h^{(1)}\left(\frac{\partial^2 v_z^{(0)}}{\partial y^2} + \frac{\partial^2 v_y^{(0)}}{\partial z\partial y}\right) + 4h_z^{(1)}\frac{\partial v_y^{(0)}}{\partial y} = 0; \tag{3.9}$$

$$z \to -\infty,\quad v_y^{(1)} \to 0,\quad v_\theta^{(1)} \to 0,\quad v_z^{(1)} \to \frac{1}{RE\,\mathrm{Fr}}\left(-\frac{y^3}{6} + \frac{y^2}{2} - \frac{y}{2}\right); \tag{3.10}$$

$$z \to \infty,\quad v_y^{(1)} \to 0,\quad v_\theta^{(1)} \to \Omega_0 R,\quad v_z^{(1)} \to \frac{1}{RE\,\mathrm{Fr}}\left(-\frac{y^3}{6} + \frac{y^3}{2} - \frac{y}{2}\right). \tag{3.11}$$

After substitution of Eq. (3.6) in boundary conditions (3.9) the latter take on the form

$$y = 1,\quad \frac{\partial v_\theta^{(1)}}{\partial y} = 0,\quad v_y^{(1)} = h_z^{(1)}\frac{1}{2E\,\mathrm{Fr}},$$
$$2E\frac{\partial v_y^{(1)}}{\partial y} + \alpha h_{zz}^{(1)} = p^{(1)},\quad \frac{\partial v_y^{(1)}}{\partial z} + \frac{\partial v_z^{(1)}}{\partial y} + h^{(1)}\left(-\frac{1}{E\,\mathrm{Fr}}\right) = 0. \tag{3.12}$$

We will now reduce system (3.7) to a single equation in sixth order partial derivatives for the radial velocity components.

After transformations we obtain

$$4\frac{\partial^2 v_y^{(1)}}{\partial z^2} = E\left\{\frac{\partial^2}{\partial z^2}\Delta\Delta v_y^{(1)} + \left(\frac{\partial^3}{\partial y^3} + \frac{\partial}{\partial y\partial z^2}\right)\Delta\frac{\partial v_y^{(1)}}{\partial y}\right\},\quad \Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{3.13}$$

Performing a Fourier transform [5] on Eq. (3.13), we obtain

$$\frac{d^6 V_y}{dy^6} - 3\xi^2\frac{d^4 V_y}{dy^4} + 3\xi^4\frac{d^2 V_y}{dy^2} + (4E^{-1}\xi^2 - \xi^6)V_y = 0, \tag{3.14}$$

$$V_y = \int_{-\infty}^{\infty} v_y^{(1)}(z, y)\exp(iz\xi)\,dz;$$

$$V_y = C_1 e^{\alpha_1 y} + C_2 e^{\alpha_2 y} + C_3 e^{\alpha_3 y} + C_4 e^{\alpha_4 y} + C_5 e^{\alpha_5 y} + C_6 e^{\alpha_6 y}, \tag{3.15}$$

$$\alpha_{1,2} = \pm\left[\xi^2 - (2E^{-1/2}\xi)^{2/3}\right]^{1/2},$$

$$\alpha_{3,4} = \{2\xi^2 + (2E^{-1/2}\xi)^{2/3} \pm \sqrt{8\xi^2(2E^{-1/2}\xi)^{2/3} - 3(2E^{-1/2}\xi)^{4/3}}\}/2,$$

$$\alpha_{5,6} = -\{2\xi^2 + (2E^{-1/2}\xi)^{2/3} \pm \sqrt{8\xi^2(2E^{-1/2}\xi)^{2/3} - 3(2E^{-1/2}\xi)^{4/3}}\}/2.$$

The constants C_i (i = 1, 6) are found from boundary conditions (3.8), (3.10)-(3.12). We write the axial and azimuthal velocity and pressure components in the form

$$v_z^{(1)} = \frac{1}{E\,\mathrm{Fr}\,R}\left(-\frac{y^3}{6} + \frac{y^2}{2} - \frac{y}{2}\right) + V_*,\quad v_\theta^{(1)} = v_\theta^* + \theta_0,\quad p^{(1)} = p_* + \vartheta,$$

where $\theta_0 = 0$, if $z < 0$; $\theta_0 = 2\Omega_0 R$, if $z > 0$; $\vartheta_0 = 0$, if $z < 0$; $\vartheta = R(y-1)\Omega_0$, if $z > 0$. Substituting these values in boundary conditions (3.8), (3.10), (3.12) and performing a Fourier transform, we obtain

$$
\begin{aligned}
y=0,\quad & V_y=0,\quad \frac{dV_y}{dy}=0,\\
& 2V_\theta=-\frac{dp_*}{dy}+E\left(\frac{d^2V_y}{dy^2}-\xi^2V_y\right)=0,\\
y=1,\quad & V_y=(-i\xi)H/(2E\,\mathrm{Fr}),\quad 2EdV_y/dy-\alpha\xi^2H=p_*,\\
& \xi^2V_y+d^2V_y/dy^2+i\xi H/(E\,\mathrm{Fr})=0,\\
& \frac{dV_\theta}{dy}=-\frac{d^2p_*}{dy^2}+E\frac{d}{dy}\left(\frac{d}{dy}V_y-\xi^2V_y\right)=0,\\
& H=\int_{-\infty}^{\infty}h^{(1)}(z)\,e^{i\xi z}dz,\quad p_*=\int_{-\infty}^{\infty}p_*\,e^{i\xi z}\,dz.
\end{aligned}
\tag{3.16}
$$

We will now eliminate p_* and H from system (3.16). From the second equation of system (3.7), with consideration of the fact that $dp^{(1)}/dz = dp_*/dz + 2\delta(z)\Omega_0R$ ($\delta(z)$ is a delta-function), and the third equation of system (3.16), it follows that

$$
p_*=\frac{2\Omega_0R}{i\xi}+\frac{E}{\xi^2}\left(\frac{d^3V_y}{dy^3}-\xi^2\frac{dV_y}{dy}\right),\quad H=2E\,\mathrm{Fr}\,V_y/(-i\xi).
\tag{3.17}
$$

Substituting Eqs. (3.17), (3.15) in system (3.16), we obtain a system of algebraic equations the solution of which has the form

$$
\begin{aligned}
C_6=&\frac{2\Omega_0R}{Ei\xi}\left\{\Phi\left[(W_3A_4+W_4)\frac{(A_5-B_5)}{(B_4-A_4)}+W_3A_5+W_5\right]+\right.\\
&\left.+\frac{A_6-B_6}{B_4-A_4}(W_3A_4+W_4)+W_3A_6+W_5\right\}^{-1},\quad C_5=\Phi C_6,\\
C_4=&(A_5-B_5)C_5/(B_4-A_4)+(A_6-B_6)C_6/(B_4-A_4),\\
C_3=&A_4C_4+A_5C_5+A_6C_6,\\
C_2=&\{(\alpha_1-\alpha_3)C_3+(\alpha_1-\alpha_4)C_4+(\alpha_1-\alpha_5)C_5+\\
&+(\alpha_1-\alpha_6)C_6\}\ (\alpha_2-\alpha_1)^{-1},\ C_1=-C_2-C_3-C_4-C_5-C_6,\\
B_i=&-\{[(\xi^2+\alpha_2-2)e^{\alpha_2}-(\xi^2+\alpha_1-2)e^{\alpha_1}](\alpha_1-\alpha_i)(\alpha_2-\alpha_1)^{-1}+\\
&+(\xi^2+\alpha_i-2)e^{\alpha_i}-(\xi^2+\alpha_1-2)e^{\alpha_1}\}\{[(\xi^2+\alpha_2-2)e^{\alpha_2}-\\
&-(\xi^2+\alpha_1-2)e^{\alpha_1}](\alpha_1-\alpha_3)(\alpha_2-\alpha_1)^{-1}+(\xi^2+\alpha_3-2)e^{\alpha_3}-(\xi^2+\alpha_1-2)e^{\alpha_1}\}^{-1},\\
A_i=&[(\alpha_2^4-\alpha_1^4)(\alpha_i-\alpha_1)(\alpha_2-\alpha_1)^{-1}+\alpha_1^4-\alpha_i^4][\alpha_3^4-\alpha_1^4+(\alpha_2^4-\alpha_1^4)(\alpha_1-\alpha_3)(\alpha_2-\alpha_1)]^{-1},\\
D_i=&\{[(\alpha_2^5-\xi^4\alpha_2)e^{\alpha_2}-(\alpha_1^5-\xi^4\alpha_1)e^{\alpha_1}](\alpha_1-\alpha_i)(\alpha_2-\alpha_1)^{-1}+\\
&+(\alpha_i^5-\xi^4\alpha_i)e^{\alpha_i}-(\alpha_1^5-\xi^4\alpha_1)e^{\alpha_1}\}\{(\alpha_3^5-\xi^4\alpha_3)e^{\alpha_3}-(\alpha_1^5-\xi^4\alpha_1)e^{\alpha_1}]+\\
&+[(\alpha_2^5-\xi^4\alpha_2)e^{\alpha_2}-(\alpha_1^5-\xi^4\alpha_1)e^{\alpha_1}](\alpha_1-\alpha_3)(\alpha_2-\alpha_1)^{-1}\}^{-1},\quad i=3,4,5,6,\\
W_i=&[(3\alpha_2-\alpha_2/\xi^2-2i\alpha\xi\,\mathrm{Fr})e^{\alpha_2}-(3\alpha_1-\alpha_1/\xi-2\alpha i\xi\,\mathrm{Fr})e^{\alpha_1}]\times\\
&\times(\alpha_1-\alpha_i)(\alpha_2-\alpha_1)+(3\alpha_i-\alpha_i/\xi-2\alpha_i\xi\,\mathrm{Fr})e^{\alpha_i}-(3\alpha_1-\alpha_1/\xi^2-2\alpha i\xi\,\mathrm{Fr})e^{\alpha_1},\quad i=3,4,5,6,\\
\Phi=&-\frac{(A_6-D_6)/(D_4-A_4)+(A_6-B_6)/(B_4-A_4)}{(A_5-D_5)/(D_4-A_4)-(A_5-B_5)/(B_4-A_4)}.
\end{aligned}
\tag{3.18}
$$

Using a reverse Fourier transform, we find $v_\lambda^{(1)}$, $p^{(1)}$, $h^{(1)}$ (λ = y, z, θ), with consideration of the fact that the expression $\sqrt{s}$ should be understood as a unique function which coincides with the arithmetic value of the root along the positive real semiaxis on the upper edge of the section. The distribution of the functions $v_y^{(1)}$, $v_z^{(1)}$, $R^{(1)}$, $v_\theta^{(1)}$ which define the reverse Fourier transform

$$
\theta(x,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{-sz}g(y,s)\,ds,\quad s=i\xi,
\tag{3.19}
$$

have different forms in the regions $z < 0$ and $z > 0$.

4. From analysis of the behavior of the integrand of Eq. (3.19) we find that at $z < 0$

$$v_z(z, y) = \frac{\varepsilon^{1/2}}{E\,\mathrm{Fr}\,R}\left(-\frac{y^3}{6} + \frac{y^2}{2} - \frac{y}{2}\right) + \frac{1}{E\,\mathrm{Fr}}\left(-\frac{y^2}{2} + y\right) + \varepsilon^{1/2}\sum_{n=1}^{N} T_n e^{\lambda_n z} R_n(y), \qquad (4.1)$$

$$v_y(z, y) = \varepsilon^{1/2}\sum_{n=1}^{N} e^{\lambda_n z} T_n R_n^0(y),$$

$$v_\theta(z, y) = \varepsilon^{1/2}\sum_{n=1}^{N} T_n \lambda_n e^{\lambda_n z} R_n^*(y) + \theta_0,$$

$$h(z) = 1 - \varepsilon^{1/2}\sum_{n=1}^{N} T_n e^{\lambda_n z} R_n(1),$$

$$R_n(y) = \sum_{k=1}^{6} C_k e^{\alpha_k(\lambda_n) y} \alpha_k(y) H_1(\lambda_n),$$

$$R_n^0(y) = \sum_{k=1}^{6} C_k e^{\alpha_k(\lambda_n) y} H_1(\lambda_n),$$

$$R_n^*(y) = \sum_{k=1}^{6} \left(\alpha_k^4(\lambda_n) + \lambda_n^4\right) C_k e^{\alpha_k(\lambda_n) y} H_1(\lambda_n),$$

$$T_n = -\lambda_n^{-1} (\partial H/\partial x)^{-1}_{x=\lambda_n},$$

$$H_1 = (W_3^* A_4^* - W_4^* \Delta)\left[(A_5^* \delta - B_5^* \Delta)\Phi + (A_6^* \delta - B_6^* \Delta)\Phi_0\right],$$

$$A_i^* = A_i \Delta, \quad B_i^* = B_i \delta, \quad D_i^* = D_i \theta,$$

$$\Phi^* = (A_6^* \delta - B_6^* \Delta)(D_4^* \Delta - A_4^* \theta) - (A_6^* \theta - D_6 \Delta)(B_4^* \Delta - A_4 \delta), \quad W_i^* = W_i \tau,$$

$$\Phi_0 = (A_5^* \theta - D_5^* \Delta)(B_4^* \Delta - A_4^* \delta) - (A_5^* \delta - B_5^* \Delta)(D_4^* \Delta - A_4^* \theta),$$

$$\Delta = (\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)\left[(\alpha_3^2 + \alpha_1^2)(\alpha_3 + \alpha_1) - (\alpha_2^2 + \alpha_1^2)(\alpha_2 + \alpha_1)\right],$$

$$\tau = x^2(\alpha_2 - \alpha_1),$$

$$\theta = \left[(\alpha_2^5 - x^4\alpha_2)e^{\alpha_2} - (\alpha_1^5 - x^4\alpha_1)e^{\alpha_1}\right](\alpha_1 - \alpha_3) + \left[(\alpha_3^5 - x^4\alpha_3)e^{\alpha_3} - (\alpha_1^5 - x^4\alpha_1)e^{\alpha_1}\right](\alpha_2 - \alpha_1),$$

$$\delta = (x^2 - \alpha_4 + 2)e^{\alpha_4}(\alpha_1 - \alpha_2) + (x^2 + \alpha_2 - 2)e^{\alpha_2}(\alpha_4 - \alpha_1) - (x^2 - \alpha_1 - 2)e^{\alpha_1}(\alpha_4 - \alpha_2),$$

$$H_1(\lambda_n) = (W_3^* A_4^* - W_4^* \Delta)\left[\Phi^*(A_5^* \delta - B_4^* \Delta) + (A_6^* \delta - B_6^* \Delta)\Phi_0\right] + (B_4^* \Delta - A_4^* \delta)\left[(W_3^* A_5^* - W_5^* \Delta)\Phi^* + \Phi_0(W_3^* A_6^* - W_6^* \Delta)\right], \quad \alpha_{1,2} = \pm\left[x^2 - (2xE^{-1/2})^{2/3}\right]^{1/2},$$

$$\alpha_{3,4} = \{2x^2 + (2xE^{-1/2})^{2/3} \pm [8x^2(2xE^{-1/2})^{2/3} - 3(2xE^{-1/2})^{4/3}]^{1/2}\}/2,$$

$$\alpha_{5,6} = -\{2x^2 + (2xE^{-1/2})^{2/3} \pm [8x^2(2xE^{-1/2})^{2/3} - 3(2xE^{-1/2})^{4/3}]^{1/2}\}/2.$$

Here W_i, A_i, B_i, C_i, D_i ($i = 3, 4, 5, 6$) are specified by Eq. (3.18), and the expressions $f(\lambda_n)$ indicate that in place of x we must substitute in the expressions the roots λ_n of the equations $H(\lambda_n) = 0$, $n = 1, 2, 3, \ldots$. Similarly, it follows from Eqs. (3.18), (3.19) that at $z > 0$ the functions v_z, v_y, v_θ, h are defined by Eq. (4.1) in which the functions α_i ($i = 1, \ldots, 6$) are replaced by the functions

$$\alpha_{1,2} = \pm[-x^2 + (2xE^{-1/2})^{2/3}]^{1/2},$$

$$\alpha_{3,4} = [-2x^2 - (2xE^{-1/2})^{2/3} \pm \sqrt{8x^2(2xE^{-1/2})^{2/3} - 3(2xE^{-1/2})^{4/3}}]/2,$$

$$\alpha_{5,6} = -[-2x^2 - (2xE^{-1/2})^{2/3} \pm \sqrt{8x^2(2xE^{-1/2})^{2/3} - 3(2xE^{-1/2})^{4/3}}]/2.$$

The functions H and roots λ_n change correspondingly. Equations (4.1), (4.2), (3.18) define the analytical solution of the problem of velocity field distribution and form of the free surface of the liquid film.

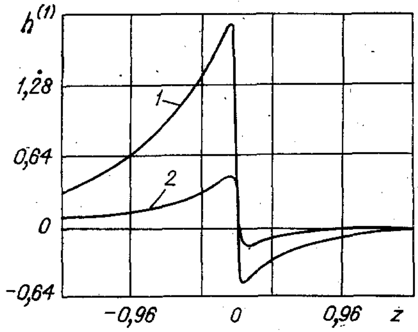

Fig. 1

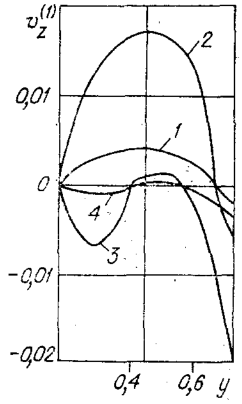

Fig. 2

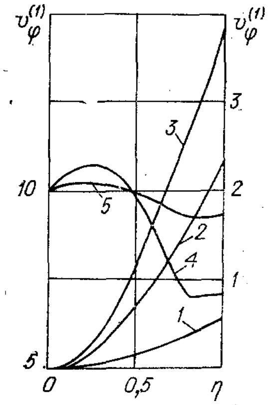

Fig. 3

5. Equations (4.1), (4.2), (3.18) were used to calculate numerically the flow of a liquid film on the inner surface of a rotating cylinder for the case where Fr = 0.6, E = 0.3, R = 5, α = 7.3, and Ω_0 takes on various values. Initially the roots λ_n of the function H were found for the region $z < 0$ as well as the roots of the corresponding functions for the region $z > 0$. The roots were found by Mueller's iteration method for solutions of high order algebraic equations [6]. After the roots were found the complex expressions (4.1), (4.2) were calculated numerically and their real components separated.

Figure 1 shows the change in film thickness along the z axis for two values of Ω_0: Curve 1 corresponds to $\Omega_0 = 2$, 2, $\Omega_0 = 0.5$. As is evident from Fig. 1, the film thickness increases rapidly in the region $z < 0$ near zero, forming a peak, then decreases abruptly, with a shallow depression appearing in the region $z > 0$. Formation of the peak is related to the fact that the pressure in the liquid is different at $z < 0$ and $z > 0$, according to Eqs. (2.7), (2.8), i.e., in the liquid near the cylinder surface at the point $z = 0$ there are pressure discontinuities, caused by the differing rotation velocities of the cylinders. If the angular velocity of rotation of the lower cylinder is greater than that of the upper, ($\Omega_2 > \Omega_1$, $\Omega_0 > 0$), then the pressure as $z \to +0$ will be higher than as $z \to -0$, i.e., at the junction there appears an additional force which will hinder liquid motion, leading to formation of the peak. The greater the discontinuity in angular velocity, the higher will be the pressure gradient, and the higher will be the peak height, which agrees with the calculation presented in Fig. 1. In the region $z > 0$ there is some acceleration of the liquid flow which leads, as follows from Fig. 1, to formation of a shallow depression. The peak and depression are found only on a small segment of the z axis of the order of one or two film thicknesses extending in both directions from the plane $z = 0$, while at other points the surface is practically undisturbed, since the pressure gradient is nonzero only near $z = 0$. Film thicknesses far from the plane $z = 0$ are identical, since in those regions the flow is determined solely by gravitational and viscosity forces which are identical.

The pressure distribution also has a sharp peak in the region $z = 0$ near the line dividing the tubes. It should be noted that such a form of pressure change has been observed in cases of other types of abrupt change in boundary conditions, for example, in flow around an isolated projection [7].

Figure 2 shows the distribution along the y axis of the third velocity term in Eq. (4.1) at $\Omega_0 = 2$ and $z = -1.62, -0.42, 0.42, 1.62$ (curves 1-4). It is evident from Fig. 2 that in the region $z < 0$ ($\Omega_0 = 2$) we must subtract from the axial velocity distribution defined by the first two terms of Eq. (4.1) a correction specified by the third term of Eq. (4.1). From this it follows that the axial velocities, according to Fig. 3, will decrease as $z \to -0$ in the range from $y = 0$ to 0.9, while in the range $0.9 \leqslant y \leqslant 1$ they will be somewhat higher than the velocities in this range at infinity. As a whole, the flow slows down, and a peak is formed at $z < 0$, which agrees with Fig. 1. At $z > 0$ the values of the correction V_* to the axial velocity are greater than zero from $0 \leqslant y \leqslant 0.4$, while at $0.4 \leqslant y \leqslant 0.72$ consideration of the correction leads to retardation of the flow, while at $0.72 \leqslant y \leqslant 1$ the correction is again positive. As is evident from Fig. 2, on the whole consideration of the third term in Eq. (4.1) leads to acceleration of the flow, which agrees with Fig. 1.

Figure 3 presents the distribution of the azimuthal component of the velocity along the y axis at $\Omega_0 = 2$ and at $z = -1.62, -0.42, -0.02, 0.82, 1.62$ (curves 1-5 respectively). It is evident from Fig. 3 that with increase in z (in the region $z < 0$) there is a continual increase in azimuthal velocity. Viscous stresses developed due to the difference in angular velocities of the semiinfinite tubes are transmitted up the flow and lead to azimuthal motion of the liquid in the range $z > 0$ near the junction. The presence of an immobile wall in the region $z < 0$ (the wall is immobile in the rotating coordinate system) hinders rotation of the liquid. Therefore the azimuthal velocity decreases upon approach to the cylinder wall. In the region $z > 0$ the liquid rotation velocity must coincide with the velocity of rotation of the semiinfinite tube; therefore with growth in z equalization of azimuthal velocity components along the layer thickness occurs, in agreement with Fig. 3.

It should be noted that the effect of "inhibiting" the flow (formation of regions of closed circulation) often found in flows like those considered here (formation of a peak on the free surface) is also found in flow of a liquid within a tube consisting of two closely joined tubes rotating with different angular velocities [8], as well as in the limiting case where one tube is at rest and the other rotates [9]. Closed circulation zones are also formed in rotating channels of variable section [10].

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